

VECTOR FIELD INDUCED CHAOS IN MULTI-DIMENSIONAL HOMOGENEOUS COSMOLOGIES

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We show that in multidimensional gravity vector fields completely determine the structure and properties of singularity. It turns out that in the presence of a vector field the oscillatory regime exists for any number of spatial dimensions and for all homogeneous models. We derive the Poincaré return map associated to the Kasner indexes and fix the rules according to which the Kasner vectors rotate. In correspondence to a 4-dimensional space time, the oscillatory regime here constructed overlap the usual Belinski-Khalatnikov-Liftshitz one.

1. Introduction

The wide interest attracted by the homogeneous cosmological models of the Bianchi classification relies over all in the allowance for their anisotropic dynamics; among them the types VIII and IX stand because of their chaotic evolution toward the initial singularity¹ that correspond to the maximum degree of generality allowed by the homogeneity constraint; as a consequence it was shown^{2–4} that the generic cosmological solution can be described properly, near the Big-Bang, in terms of the homogeneous chaotic dynamics as referred to each cosmological horizon. However the correspondence existing between the homogeneous dynamics and the generic inhomogeneous one holds only in four space-time dimensions. In fact a generic cosmological inhomogeneous model remains characterized by chaos near the Big-Bang up to a ten dimensional space-time^{5–7} while the homogeneous models show a regular (chaos free) dynamics beyond four dimensions.^{8,9}

Here we address an Hamiltonian point of view showing how the homogeneous models (of each type) perform, near the singularity, an oscillatory regime in correspondence to any number of dimensions, as soon as an electromagnetic field is included in the dynamics.

2. The Standard Kasner Dynamics

Let us consider the standard $n + 1$ -dimensional vector-tensor theory in the ADM representation:

$$I = \int d^n x dt \left\{ \Pi^{\alpha\beta} \frac{\partial}{\partial t} g_{\alpha\beta} + \pi^\alpha \frac{\partial}{\partial t} A_\alpha + \varphi D_\alpha \pi^\alpha - N H_0 - N^\alpha H_\alpha \right\}, \quad (1)$$

$$H_0 = \frac{1}{\sqrt{g}} \left\{ \Pi_\beta^\alpha \Pi_\alpha^\beta - \frac{1}{n-1} (\Pi_\alpha^\alpha)^2 + \frac{1}{2} g_{\alpha\beta} \pi^\alpha \pi^\beta + g \left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - R \right) \right\}, \quad (2)$$

$$H_\alpha = -\nabla_\beta \Pi_\alpha^\beta + \pi^\beta F_{\alpha\beta}, \quad (3)$$

Here H_0 and H_α denote respectively the super-Hamiltonian and super-momentum, $F_{\alpha\beta} \equiv \partial_\beta A_\alpha - \partial_\alpha A_\beta$ is the electromagnetic tensor, $g \equiv \det(g_{\alpha\beta})$ is the determinant of the n-metric, R is the n-scalar of curvature and $D_\alpha \equiv \partial_\alpha + A_\alpha$.

Since the sources are absent, it is enough to consider only the transverse components for A_α and π^α ; therefore, we take the gauge conditions $\varphi = 0$ and $D_\alpha \pi^\alpha = 0$. When going over the homogeneous case, we choose the gauge $N = 1$ and $N^\alpha = 0$.

Let's adopt the Kasner parameterization, that is based on the metric and conjugate momentum decomposition along spatial n-bein:

$$g_{\alpha\beta} = \delta_{ab} l_\alpha^a l_\beta^b, \quad \Pi_{\alpha\beta} = p_{ab} l_\alpha^a l_\beta^b, \quad (4)$$

We also define a dual basis $L_a^\alpha = g^{\alpha\beta} l_\beta^a$, such that $L_a^\alpha l_\alpha^b = \delta_a^b$ and $L_a^\alpha l_\beta^a = \delta_\beta^\alpha$.

We want to put in evidence the oscillatory regime that the bein vectors possess and so we distinguish scale functions and the parallel from the transverse component ($\tilde{\lambda}_a = (\pi^\alpha \ell_\alpha^a)$)

$$l_a = \exp(q^a/2) \ell_a, \quad L_a = \exp(-q^a/2) \mathcal{L}_a. \quad (5)$$

$$\vec{\ell}_a = \vec{\ell}_{a\parallel} + \vec{\ell}_{a\perp}; \quad \vec{\ell}_{a\parallel} = \frac{\tilde{\lambda}_a}{\pi^2} \vec{\pi}, \quad (\vec{\pi} \vec{\ell}_{a\perp}) = 0. \quad (6)$$

The standard Kasner solution is obtained as soon as the limit in which all the terms $\exp(q^a)$ become of higher order is taken

$$\begin{aligned} p_a &= \text{const}, & \tilde{\lambda}_a &= \text{const}, & \vec{\ell}_{a\perp} &= \text{const}, \\ \frac{\partial}{\partial t} q_a &= \frac{2N}{\sqrt{g}} \left(p_a - \frac{1}{n-1} \sum_b p_b \right), \\ \sum p_a^2 - \frac{1}{n-1} (\sum p_a)^2 + \frac{1}{2} \sum e^{q_a} \tilde{\lambda}_a^2 &= 0, \end{aligned} \quad (7)$$

$$g_{\alpha\beta} = \sum_a t^{2s_a} \ell_\alpha^a \ell_\beta^a, \quad s_a = 1 - (n-1) \frac{p_a}{\sum_b p_b}, \quad (8)$$

The Kasner indexes s_a satisfy the identities $\sum s_a = \sum s_a^2 = 1$.

3. Billiard representation: the return map and the rotation of Kasner vectors

If we order the s_a 's, the largest increasing term (as $t \rightarrow 0$ $t^{s_1} \rightarrow \infty$) among the neglected ones comes from s_1 and it is to be taken into account to construct the oscillatory regime toward the cosmological singularity.

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{\lambda}_1 &= 0, & \frac{\partial}{\partial t} \tilde{\lambda}_a &= \frac{(\frac{\partial}{\partial t} p_1) \tilde{\lambda}_a}{(p_a - p_1)} \\ \frac{\partial}{\partial t} p_1 &= -\frac{N}{2\sqrt{g}} \tilde{\lambda}_1^2 \exp(q^1), & \frac{\partial}{\partial t} p_a &= 0, \\ \frac{\partial}{\partial t} q_a &= \frac{2N}{\sqrt{g}} \left(p_a - \frac{1}{n-1} \sum_b p_b \right). \end{aligned} \quad (9)$$

The first of equations (9) gives $\tilde{\lambda}_1 = \text{const}$, while the second admits the solution

$$\tilde{\lambda}_a (p_a - p_1) = \text{const}. \quad (10)$$

The remaining part of the dynamical system allows us to determine the return map governing the replacements of Kasner epochs and the rotation of Kasner vectors $\vec{\ell}_a$ through these epochs

$$s'_1 = \frac{-s_1}{1 + \frac{2}{n-2} s_1}, \quad s'_a = \frac{s_a + \frac{2}{n-2} s_1}{1 + \frac{2}{n-2} s_1}, \quad (11)$$

$$\tilde{\lambda}'_1 = \tilde{\lambda}_1, \quad \tilde{\lambda}'_a = \tilde{\lambda}_a \left(1 - 2 \frac{(n-1) s_1}{(n-2) s_a + n s_1} \right), \quad (12)$$

$$\vec{\ell}'_a = \vec{\ell}_a + \sigma_a \vec{\ell}_1, \quad \sigma_a = \frac{\tilde{\lambda}'_a - \tilde{\lambda}_a}{\tilde{\lambda}_1} = -2 \frac{(n-1) s_1}{(n-2) s_a + n s_1} \frac{\tilde{\lambda}_a}{\tilde{\lambda}_1}. \quad (13)$$

Thus the homogeneous Universes here discussed approaches the initial singularity being described by a metric tensor with oscillating scale factors and rotating Kasner vectors. The presence of a vector field is crucial because, independently on the considered model, it induces a closed domain on the configuration space.

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